

A New Approach to Modelling Strongly Non-Equilibrium , Time-Dependent Turbulent Flows

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The hydrodynamic equations can be derived from Newtonian mechanics using a coarse-graining or model building procedure. Namely, averaging over the so called “fast degrees of freedom” with the characteristic time scales $\tau_\lambda \approx \lambda/c_s$ where c_s is the speed of sound and λ is the mean free path leads to the equations for the hydrodynamic or “slow” modes that we are usually interested in. Under normal conditions, a typical microscopic time scale in gas is $\tau_\lambda \approx \frac{10^{-5}cm}{10^5cm/sec} \approx 10^{-10}sec$. The validity criterion for the coarse-graining procedure is that $\tau_\lambda \ll \tau_h = L/U$ where L and U are the typical length scale and speed of the slow (hydrodynamic) modes. This means that $\frac{\lambda}{L}Ma \ll 1$ where $Ma = \frac{U}{c_s}$ is the Mach number. Since in all low Mach number laminar flows of Newtonian fluids this criterion is well satisfied, the hydrodynamic (Navier-Stokes) equations are the most reasonable/practical way to analyze macroscopic flow behavior.

There exists, however, an intermediate way to describe statistical systems of this type. Introducing a distribution function in the configuration space, $f(\mathbf{x}, \mathbf{v}, \mathbf{c})$, where \mathbf{v} and \mathbf{c} describe “hydrodynamic” (slow) and fast (molecular) motions, respectively, one can, in principle, write down an exact Boltzmann equation with all possible interactions among all the particles accounted for. In the most general form, this involves an unspecified collision integral, C :

$$\frac{\partial f}{\partial t} + (\mathbf{v} + \mathbf{c}) \cdot \nabla f = C \quad (1)$$

Due to very large scale separation between the fast and slow modes, we can assume that

$$f(\mathbf{v}, \mathbf{c}) \approx f(\mathbf{v})\phi(\mathbf{c}). \quad (2)$$

In some cases the collision integral can be represented in a simple, quasilinear form using the relaxation time (the so called “BGK” [?]) approximation

$$C \approx -\frac{f - f^{eq}}{\tau_\lambda} \quad (3)$$

where f^{eq} is an equilibrium distribution function. This way, the information about particle dynamics is hidden in the time of relaxation to local equilibrium, τ_λ , which was estimated above. Derivation of hydrodynamic equations for the slow modes can be made exact if, and just if, several physical assumptions are made. First, there should be a well defined scale separation between fast and slow modes. Otherwise, the high order contributions to the Navier-Stokes equation are non-negligible. Secondly, we should assume that at the very small scales the collision integral satisfies the **local** conservation laws. Both these assumptions are well satisfied for a gas system involving normal hydrodynamics scales and thus the Navier-Stokes equations are justified for description of both turbulent and laminar flows. On the other hand, the huge number of degrees of freedom in a turbulent flow make full direct numerical simulation (DNS) of turbulence impossible. Viewing from our starting point, the kinetic theory, turbulence modeling responds to the need for some additional coarse-graining which is done on the next (the hydrodynamic) level.

Treating small-scale turbulent structures similar to gas molecules, conventional turbulence modeling derives effective transport coefficients in terms of the properties of small-scale fluctuations. The effective viscosity $\nu_T \propto k\tau_T$ is given in terms of

the local value of turbulent kinetic energy, $k = \langle u^2 \rangle / 2$, and the relaxation time to a local equilibrium for large scales, τ_T . The notion of turbulent kinetic energy k is similar to local temperature in conventional kinetic theory and the value of τ_T reflects the dynamics of the system, including external fields. For homogeneous isotropic turbulence, this relaxation time must scale as $\tau_T \approx k/\epsilon$, where $\epsilon \approx \nu \langle (\partial_i v_j)^2 \rangle$ is the local mean rate of kinetic energy dissipation. In this case, the effective viscosity in the fully developed turbulent regime is given by $\nu_T \propto k^2/\epsilon$, which is literally used in all the well-known $k - \epsilon$ turbulence models and implicitly used in other conventional closure models where it is sometimes recast in equivalent forms based on different choices of main dynamic variables. On the other hand, if in the large scale flow there is a strong shear, swirl, or some other anisotropy factor, the relaxation time τ_T should also include the characteristic mean flow time scales such as S_{ij} , Ω_{ij} , etc ...

It has been recognized from the very beginning of turbulence modeling activity that simple models of the $k - \epsilon$ type fail for non-equilibrium, strongly sheared, and time-dependent flows (below we present a quantitative criterion). There exist also very serious problems with the Reynolds stress and/or non-linear viscosity models. First of all, they involve many differential equations and require elaborate sets of boundary/initial conditions which makes the problem stiff and sometimes even ill-posed. On the more fundamental level, if the shear is strong or the flow is strongly non-equilibrium, there is no a priori reason to use only the finite orders in the infinite expansion series in powers of the $O(1)$ dimensionless shear (more details are also given below), even though certain regularizations via Pade approximation can be made. As long as the main argument for these complex models is that they correctly account for the $O(1)$ anisotropy effects, this does not seem logically consistent.

Since, unlike the traditional kinetic theory, there is no well defined scale separation between the slow mean flow modes and the fast fluctuating turbulent modes, the effective equation of motion for slow modes averaged over small scales has in effect contributions from all orders. Its most general form can be written down as follows:

$$(\partial_t + U_i \partial_i) \mathbf{U} + \sum_n \alpha_n \tau_T^{n-1} (\partial_t + U_i \partial_i)^n \mathbf{U} = -\nabla p + \partial_i (\nu_0 + \nu_T(k, \epsilon, S_{ij}, \dots)) \partial_i \mathbf{U} + O(\partial_i^{2n} \mathbf{U}) \quad (4)$$

where the relaxation time τ_T is a complicated functional of all parameters of the problem. Since $\nu_T \approx \langle u^2 \rangle \tau_T$, it is clear that Equation (4) can not be written in a

simple finite general form which is also invariant under Galilean, translational, and other transformations. If, however, both velocity gradients and time -derivatives are small ($\tau_T |S_{ij}| \ll 1$; $T \partial_t \mathbf{v} \geq 1$), the Equation (4) reduces to the Navier-Stokes equations-based $K - \epsilon$ model. But under general circumstances more terms beyond the Navier-Stokes based level must be retained. On the other hand, Equation (4) is very complex and even poorly defined since we do not know where to truncate an infinite series when the effects of time-dependence and strain-induced anisotropy are strong. It is easy to see that the simple $K - \epsilon$ approximation breaks down when $(\mathbf{U} \cdot \nabla) \mathbf{U} \sim \tau_T (\mathbf{U} \cdot \nabla)^2 \mathbf{U}$ which leads to $\eta = \tau_T |S_{ij}| \approx \frac{k |S_{ij}|}{\epsilon} = O(1)$. This is a typical situation in many flows of interest. Therefore, the low-order modeling of strongly non-equilibrium flows of general nature seems to be simply impossible. What one needs is a smart resummation of the expansion (4) which we cannot even write down explicitly.

There is another opportunity, though. Taking a step back, we find a subtle inconsistency in the conventional turbulence modeling ideology: postulating the existence of the relaxation time $\tau_T(\mathbf{x}, t) \approx \frac{k}{\epsilon}$ we tacitly assume that the gas of eddies, if exists, is still described by some Navier-Stokes equation with just a modified viscosity. This runs into trouble as described above. On the other hand, from the above discussion we see that such a gas of eddies is adequately describable by a corresponding Boltzmann equation with the collision integral C^1 written in the relaxation time approximation, where now τ_T is a time/space-dependent relaxation time corresponding to slow modes. One very important point is to be stressed: the probability density $f(\mathbf{x}, \mathbf{v}, t)$ now describes the gas of “turbulence particles” which are slow, large, and variable-shape. That is why the collision integral C^1 satisfies all **global** conservation laws but violates the **local** ones. When we trace the derivation of hydrodynamics from the Boltzmann equation, the above mentioned failure seems to be only associated with the Navier-Stokes (or any other finite-order) truncations, but not with the Boltzmann representation itself. Hence, the fundamental turbulence modeling idea of viewing eddies as a gas system is much better justified at the Boltzmann level, involving an appropriate relaxation time. This way, the $O(S_{ij}^2)$ non-linear tensorial correction to the $k - \epsilon$ eddy viscosity is generated merely by an expansion of the collision integral in the “effective Boltzmann equation”. Higher order corrections can

be also generated by the same formal expansion. If, based on experimental data or on some theoretical arguments like renormalization group-based analysis, we fix the first few coefficients to give a correct equation for equilibrium isotropic turbulent flow, then the Boltzmann equation with $\tau_T = \tau_T(\mathbf{x}, k/\epsilon, S_{ij}, \dots)$ must correspond to at least a partial resummation of the infinite expansion (4). This model is constructed analogously to (1)-(3) with the exception that $\lambda \rightarrow L$ and $\tau_\lambda \rightarrow \tau_T$ where $\tau_T = \tau(\mathbf{x}, k/\epsilon, S_{ij}, \dots)$ and L is the fluctuation (integral) scale of turbulence. The $K - \epsilon$ equations are obtained if L^2/T is space-time independent and there is a well defined scale separation in the turbulent field. Consequently, our best (perhaps the only) hope for description of strongly non-equilibrium flows is to show that a Boltzmann equation with the properly given relaxation time is a good approximation to the entire expansion (4). **In other words, we would like to blame (at least in part) the deficiencies of the existing turbulence models applied to strongly non-equilibrium and strongly time-dependent flows not on our poor understanding of the basic physics, but rather on an unjustified expansion and truncation of (4).**

A case is being made here that our understanding of physics of turbulence is good and complete enough to enable us to describe complex 3D flows. The problem is that all models which incorporate the Navier-Stokes equations with transport coefficients obtained from finite (low) order expansions in powers of large dimensionless strain, swirl, helicity,... are as bad and poorly defined as the expansions they are based upon. Thus, the only way to quantitatively describe strongly non-equilibrium flows is to find transport functions from which, in fact, these expansions have originated.

Our approach based on a Boltzmann equation constructed in such a way that at least in the leading orders it produces the well-known, well-tested turbulence models contains the correct subset of the entire infinite expansion which routinely arises in formal derivation of general turbulence models. The kinetic equation for mean flow is combined in our approach with two transport equations for the K and ϵ scalars. At the first glance, these equations are simple and no new physics seems to have been introduced. Let us realize, however, that both K and ϵ are functionals of the velocity field \mathbf{v} , strain s_{ij} , etc obtained from the equation of motion which is quite different from the Navier-Stokes-based models. What is implicitly assumed in our new model

is that whereas the basic dynamic variable is a vector field \mathbf{v} which is responsible for the non-trivial tensorial structure of the equations, the turbulent energy/dissipation rate are the secondary scalar variables of much less pronounced dynamical content. Therefore, these scalars are more likely to be efficiently described at the lowest order in the anisotropy parameters like η . There is another point: the K and ϵ values now determine the relaxation time τ_T , instead of the eddy-viscosity. As discussed in the Introduction, the latter two characteristics are only interchangeable in the limit of isotropic turbulent flows involving zero strains on the large scales, and beyond this limit τ_T is a much more meaningful physical quantity. However, even this simplified description has been sufficient to predict many essential features of very complex high Reynolds number non-equilibrium flows.